An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXXII, 2026, f. 2
https://doi.org/10.47743/anstim.2026.00017

Remarks on symmetric bi-semiderivations and semiprime ideals of rings

Merve Kılıçer · Öznur Gölbaşı

Abstract The well-known result of E. C. Posner state that a prime ring, if the commutator of each element and its image under a nonzero derivation is central, then the ring is commutative. This result has played a fundamental role in the development of the theory of centralizing mappings on noncommutative rings. Our purpose of this paper is to show this result and its generalizations to symmetric bi-semiderivation in rings without imposing any conditions on the ring, only containing a semiprime ideal. Specifically, we investigate rings that contain a semiprime ideal and prove that centralizing or commuting conditions imposed on symmetric bi-semiderivations force strong structural constraints.

Keywords derivations · symmetric bi-derivations · commutativity · semiderivations · semiprime ideals

Mathematics Subject Classification (2010) 16W25 · 16N60 · 16U80

1 Introduction

Let R will be an associative ring. A proper ideal P of R is said to be prime if for any $x, y \in R, xRy \subseteq P$ implies that $x \in P$ or $y \in P$. The ring R is prime precisely when its zero ideal is prime; equivalently for $x, y \in R$ the equality $xRy = (0)$ implies $x = 0$ or $y = 0$. Recall that a proper ideal P of R is said to be semiprime if for any $x \in R, xRx \subseteq P$ implies that $x \in P$. A ring is semiprime whenever its zero ideal is semiprime. Every prime ideal is semiprime ideal but the converse is not true in general.

Derivations form a central theme in ring theory. An additive mapping $d : R \to R$ is a derivation if it satisfies the Leibniz identity $d(xy) = d(x)y+xd(y)$ for all $x, y \in R$. The structural consequences of derivations on prime and semiprime rings have been extensively studied, particularly following Posner's seminal result asserting that a prime ring admitting a central derivation must be commutative in [9]. Many mathematicians have studied the properties of derivations in various types of non-commutative rings, prime rings, semiprime rings, near-rings and algebras. In 1983, J. Bergen [4] has

been defined the concept of semiderivation of ring as follows: An additive mapping d on R is called a semiderivation of ring if there exist a map g on R such that (i) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and (ii) $d(g(x)) = g(d(x))$, for all $x, y \in R$. In this definition if g is the identity map on R, the concept of semiderivation covers the concept of derivation. The most natural example of semiderivations, we consider a homorphism g on a ring R such that $g \neq 1$ and define a map as $f = g - 1$. With such construction the map f a semiderivation which is not a derivation.

The concept of symmetric bi-derivation was first introduced by Maksa in [7]. A mapping $D(.,.) : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $d : R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x, x)$ for all $x \in R$. It is obvious that if $D(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x + y) = d(x) + d(y) + 2D(x, y)$, for all $x, y \in R$. If $D(.,.)$ is bi-additive and satisfies the identities

$$D(xy, z) = D(x, z)y + xD(y, z)$$

and

$$D(x, yz) = D(x, y)z + yD(x, z),$$

for all $x, y, z \in R$. Then $D(.,.)$ is called a symmetric bi-derivation. According to [13], symmetric bi-additive function $D(.,.) : R \times R \to R$ is called a symmetric bi-semiderivation associated with a function $f : R \to R$ if

$$D(xy, z) = D(x, z)f(y) + xD(y, z) = D(x, z)y + f(x)D(y, z)$$

$$D(x, yz) = D(x, y)f(z) + yD(x, z) = D(x, y)z + f(y)D(x, z),$$

and $D(f(x)) = f(D(x))$ for all $x, y, z \in R$. Yılmaz et al. obtain some well-known results symmetric bi-semiderivations on prime rings. This definition unifies the notions of symmetric bi-derivation and covers the concepts of derivations, semiderivations.

2 Example

[10, Example 1.1] Suppose the ring R is commutative ring and $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in R \right\}$. Define maps $D : R \times R \to R$ as follows:

$$D\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & bd \\ 0 & 0 \end{pmatrix}\right)$$

and $d : R \to R$ by

$$d\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)$$

Then it is easy to verify that D is a symmetric bi-semiderivation on R associated with d.

For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the commutator $xy + yx$. Let S be a nonempty subset of R. A mapping F from R to R is called commuting on S if $[F(x), x] = 0$, for all $x \in S$. This definition has been generalized such as: A map $F : R \to R$ is called a $U-$commuting map on S if $[F(x), x] \in U$, for all $x \in S$ and some $U \subseteq R$. In particular, if $U = 0$, then F is called a commuting map on S. Note that every commuting map is a $U-$commuting map. But the converse is not true in general.

The first study on commuting maps was made by N. Divinsky in 1955 [5]. This result of N. Divinsky was generalized by J. Luh and was considered for arbitrary prime rings [6]. J. H. Mayne proved that the ring is commutative by taking central automorphism on a prime ring [8]. In 1957, E. C. Posner proved that a prime ring with central derivation is commutative [9]. This theorem is also called Posner's Second Theorem. Subsequently, many authors generalized this theorem by considering suitable subsets of rings or various generalizations of derivations, such as semiderivations and generalized derivations.

In [12], Vukman investigated symmetric bi-derivations with centralizing mappings and obtained notable results for prime rings. Later, concept of generalized symmetric bi-derivations was introduced in [2] by Argaç and Shujat was proved some results concerning symmetric generalized derivations on prime and semiprime rings in [10]. In [13], the authors investigated the relationship between symmetric bi-semiderivations and Lie ideals of a prime ring. Bedir proved some results concerning symmetric bi-derivations in rings to two-sided ideals of semiprime rings in [3].

In recent years, several results established for prime or semiprime rings with derivations have been extended to arbitrary rings containing prime or semiprime ideals, with structural properties derived via the factor ring R/P. In [11], the authors investigated symmetric bi-derivations with centralizing mappings and obtained notable results for prime ideals of any ring.

The principal aim of this paper is to investigate the structural impact of symmetric bi-semiderivations on rings that contain a semiprime ideal, without assuming that the entire ring is prime or semiprime. We demonstrate that centralizing or commuting symmetric bi-semiderivations acting on a semiprime ideal impose strong annihilator conditions and, under appropriate assumptions, force commutativity in the corresponding factor structures. Several known Posner-type results arise as special cases of our general framework. To the best of our knowledge, this is one of the first systematic studies examining the interaction between semiprime ideals and symmetric bi-semiderivations under centralizing conditions.

In this paper, we shall frequently use the following identities:

$[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$

$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$

$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z]$.

3 Results

Lemma 3.1 [1, *Lemma* 1] *Let R be a ring with P a semiprime ideal of R, I an ideal of R, and $a \in I$ such that $axa \in P$ for all $x \in I$, then $a \in P$.*

Theorem 3.2 *Let R be a ring, I a nonzero ideal and P a semiprime ideal of R. If $char(R/P) \neq 2$ and $D : R \times R \to R$ a symmetric bi-semiderivation associated with a function $f : R \to R$, d be the trace of D which satisfies $d(x) \in P$ and $D(R, R) \subset I$, for all $x \in I$, then $D(R, R) \subseteq P$.*

Proof. By assumption, we have

$$d(x) \in P, \text{ for all } x \in I.$$

Substituting $x + y$ in place of x in the above relation yields

$$d(x) + d(y) + 2D(x, y) \in P, \text{ for all } x, y \in I.$$

Using $d(x), d(y) \in P$ and $char(R/P) \neq 2$, we obtain that

$$D(x, y) \in P, \text{ for all } x, y \in I.$$

Next, replacing x by xr where $r \in R$ in the above expression and using last expression, we have

$$xD(r, y) \in P, \text{ for all } x, y \in I, r \in R.$$

Equivalently,

$$ID(r, y) \subseteq P, \text{ for all } y \in I, r \in R,$$

Applying Lemma 3.1, we deduce that

$$D(r, y) \in P, \text{ for all } y \in I, r \in R.$$

Now, writing y by $ys, s \in R$ in this expression and using the previous conclusion, we derive

$$yD(r, s) \in P, \text{ for all } y \in I, r, s \in R.$$

A further application of Lemma 3.1 yields $D(r, s) \in P$, for all $r, s \in R$. This completes the proof.

□

Theorem 3.3 *Let R be a ring, I a nonzero ideal and P a semiprime ideal of R. If $char(R/P) \neq 2$ and $D : R \times R \to R$ a symmetric bi-semiderivation associated with a function $f : R \to R$, d be the trace of D which satisfies $D(xoy, z) \in P$, for all $x, y, z \in I$, then d is $P-$commuting map on I.*

Proof. We have

$$D\left(xoy,z\right)\in P \text{ for all } x,y,z\in I. \tag{3.1}$$

Replacing yx instead of y in (3.1) and expanding this, we get

$$D\left(xoyx,z\right)=D(xoy,z)f(x)+(xoy)D(x,z)\in P \text{ for all } x,y,z\in I.$$

Using our hypothesis, we find that

$$(xoy)D(x,z)\in P \text{ for all } x,y,z\in I. \tag{3.2}$$

Writing x instead of z in (3.2), we arrive at

$$(xoy)d(x)\in P \text{ for all } x,y\in I. \tag{3.3}$$

Taking y by $ry,r\in R$ in (3.3) and using this, we have

$$[x,r]yd(x)\in P \text{ for all } x,y\in I,r\in R. \tag{3.4}$$

Replacing yx instead of y in (3.4), we get

$$[x,r]\,yxd\left(x\right)\in P \text{ for all } x,y\in I,r\in R. \tag{3.5}$$

On the other hand, multiplying expression (3.4) on the right for x, we have

$$[x,r]\,yd\left(x\right)x\in P \text{ for all } x,y\in I,r\in R. \tag{3.6}$$

By comparing (3.5) and (3.6) relations, we find that

$$[x,r]\,y[d\left(x\right),x]\in P \text{ for all } x,y\in I,r\in R.$$

Writing $d\left(x\right)$ instead of r in last expression, we get

$$[d\left(x\right),x]\,y\,[d\left(x\right),x]\in P \text{ for all } x,y\in I.$$

It follows that

$$[d\left(x\right),x]\,I\,[d\left(x\right),x]\in P \text{ for all } x\in I.$$

By Lemma 3.1, we arrive at $[d\left(x\right),x]\in P$ for all $x\in I$. Hence d is $P-$commuting map on I.

□

Theorem 3.4 *Let R be a ring, I a nonzero ideal and P a semiprime ideal of R. If $char(R/P)\neq 2$ and $D:R\times R\to R$ a symmetric bi-semiderivation associated with a function $f:R\to R$, d be the trace of D which satisfies $D\left([x,y],z\right)\in P$, for all $x,y,z\in I$, then d is $P-$commuting map on I.*

Proof. By the hypothesis, we have

$$D([x,y],z) \in P \text{ for all } x,y,z \in I. \tag{3.7}$$

Replacing yx by y in (3.7), we get

$$D([x,yx],z) = D([x,y]x,z) = D([x,y],z)f(x) + [x,y]D(x,z) \in P.$$

By our hypothesis, we obtain that

$$[x,y]D(x,z) \in P \text{ for all } x,y,z \in I. \tag{3.8}$$

Writing x instead of z in (3.8), we arrive at

$$[x,y]d(x) \in P \text{ for all } x,y \in I. \tag{3.9}$$

Taking ry by $y, r \in R$ in (3.9) and using this, we have

$$[x,r]yd(x) \in P \text{ for all } x,y,r \in I.$$

This expression coincides with (3.4) in the proof of Theorem 3.3. Applying the same reasoning as in that proof, we immediately obtain the desired result.

□

Lemma 3.5 *Let R be a ring, I a nonzero ideal and P a semiprime ideal of R. If $char(R/P) \neq 2$ and $D : R \times R \to R$ a symmetric bi-semiderivation associated with a function $f : R \to R$, d be the trace of D which satisfies $d(x) \circ y \in P$, for all $x, y \in I$, then d is $P-$commuting map on I.*

Proof. By the hypothesis, we get

$$d(x) \circ y \in P, \text{ for all } x,y \in I.$$

Taking y by yz in the last expression, we obtain that

$$y(d(x) \circ z) + [d(x),y]z \in P, \text{ for all } x,y,z \in I.$$

Using the hypothesis, we have

$$[d(x),y]z \in P, \text{ for all } x,y,z \in I.$$

By Lemma 3.1, we conclude that

$$[d(x),y] \in P, \text{ for all } x,y \in I.$$

Replacing y by x in the last expression, we get

$$[d(x),x] \in P, \text{ for all } x \in I.$$

Hence we conclude that d is $P-$commuting on I.

□

Theorem 3.6 *Let R be a ring, I a nonzero ideal and P a semiprime ideal of R, $char(R/P) \neq 2$ and $D : R \times R \to R$ a symmetric bi-semiderivation associated with a function $f : R \to R$, d be the trace of D. If any of the following conditions is satisfied for all $x, y \in I$, then d is $P-$commuting map on I.*
i) $d(x \circ y) \pm (d(x) \circ y) \in P$,
ii) $d([x, y]) \pm (d(x) \circ y) \in P$,
iii) $d(x \circ y) \pm [d(x), y] \in P$,
iv) $d(x) \circ y \pm [d(y), x] \in P$,

Proof. i) By the hypothesis, we get

$$d(x \circ y) \pm (d(x) \circ y) \in P, \text{ for all } x, y \in I.$$

Taking y by $y + z, z \in I$, we have

$$d(x \circ y) + d(x \circ z) + 2D(x \circ y, x \circ z) \pm d(x) \circ y \pm d(x) \circ z \in P.$$

Using the hypothesis, we arrive at

$$2D(x \circ y, x \circ z) \in P.$$

Since $char(R/P) \neq 2$, we have

$$D(x \circ y, x \circ z) \in P.$$

Replacing z by y in this expression, we find that

$$D(x \circ y, x \circ y) \in P, \text{ for all } x, y \in I$$

and so

$$d(x \circ y) \in P, \text{ for all } x, y \in I.$$

Using this in our hypothesis, we obtain that

$$d(x) \circ y \in P, \text{ for all } x, y \in I.$$

We see that d is $P-$commuting map on I by Lemma 3.5.
ii) By the hypothesis, we have

$$d([x, y]) \pm (d(x) \circ y) \in P, \text{ for all } x, y \in I.$$

Taking y by $y + z, z \in I$ in the hypothesis, we get

$$d([x, y]) + d([x, z]) + 2D([x, y], [x, z]) \pm d(x) \circ y \pm d(x) \circ z \in P.$$

Using the hypothesis and $char(R/P) \neq 2$, we find that

$$D([x, y], [x, z]) \in P.$$

Replacing z by y in the last expression, we see that

$$D([x, y], [x, y]) \in P, \text{ for all } x, y \in I.$$

That is

$$d([x,y]) \in P, \text{ for all } x,y \in I.$$

Using this expression in our hypothesis, we get

$$d(x) \circ y \in P, \text{ for all } x,y \in I.$$

By Lemma 3.5, we obtain that d is $P-$commuting on I.

iii) By the hypothesis, we have

$$d(x \circ y) \pm [d(x), y] \in P, \text{ for all } x,y \in I.$$

Taking y by $y+z, z \in I$, we get

$$d(x \circ y) + d(x \circ z) + 2D(x \circ y, x \circ z) \pm [d(x), y] \pm [d(x), z] \in P.$$

Using the hypothesis, we see that

$$D(x \circ y, x \circ z) \in P.$$

Replacing z by y in the last expression, we get

$$D(x \circ y, x \circ y) \in P, \text{ for all } x,y \in I$$

and so

$$d(x \circ y) \in P, \text{ for all } x,y \in I.$$

Using this in the hypothesis, we have

$$[d(x), y] \in P, \text{ for all } x,y \in I.$$

Replacing y by x in the last expression, we obtain that d is $P-$commuting on I.

iv) We have

$$d(x) \circ y \pm [d(y), x] \in P, \text{ for all } x,y \in I.$$

Replacing y by $y+z, z \in I$, we get

$$d(x) \circ y + d(x) \circ z \pm 2[D(y,z), x] \pm [d(y), x] \pm [d(z), x] \in P.$$

By the hypothesis, we have

$$2[D(y,z), x] \in P.$$

Using $char(R/P) \neq 2$, we find that

$$[D(y,z), x] \in P.$$

Replacing z by y in this expression, we get

$$[D(y,y), x] \in P, \text{ for all } x,y \in I$$

and so

$$[d(y), x] \in P, \text{ for all } x,y \in I.$$

Using this in our hypothesis, we have

$$d(x) \circ y \in P, \text{ for all } x,y \in I.$$

By Lemma 3.5, we conclude that d is $P-$commuting on I.

□

Theorem 3.7 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R. If $char(R/P) \neq 2$ and $D : R \times R \to R$ a symmetric bi-semiderivation associated with a function $f : R \to R$, d be the trace of D which satisfies $d(xy) - d(x)y - xd(y) \in P$, for all $x, y, z \in I$, then d is $P-$commuting map on I.*

Proof. By the hypothesis, we get

$$d(xy) - d(x)y - xd(y) \in P \text{ for all } x, y \in I.$$

Taking y by $y + z, z \in I$ in the hypothesis, we see that

$$d(xy) + d(zy) + 2D(xy, zy) - d(x)y - d(z)y - 2D(x, z)y - xd(y) - zd(y) \in P.$$

Using the hypothesis and $char(R/P) \neq 2$, we have

$$D(xy, zy) - D(x, z)y \in P \text{ for all } x, y, z \in I.$$

Replacing z by y, we write

$$d(xy) - d(x)y \in P \text{ for all } x, y \in I. \tag{3.10}$$

Again writing y by $y + z, z \in I$ in the last expression, we get

$$d(xy) + d(xz) + 2D(xy, xz) - d(x)y - d(x)z \in P$$

and so

$$2D(xy, xz) \in P.$$

Since $char(R/P) \neq 2$, we have

$$D(xy, xz) \in P \text{ for all } x, y, z \in I.$$

Again taking z by x in this expression, we find that

$$d(xy) \in P \text{ for all } x, y \in I.$$

Using this expression in (3.10), we arrive at

$$d(x)y \in P \text{ for all } x, y \in I.$$

In particular, we have

$$d(x)x \in P \text{ for all } x \in I. \tag{3.11}$$

On the other hand, using (3.11) and $d(xy) \in P$ in our hypothesis, we get

$$xd(y) \in P \text{ for all } x, y \in I$$

and so

$$xd(x) \in P \text{ for all } x \in I. \tag{3.12}$$

Hence we obtain that $[d(x), x] \in P$, for all $x \in I$ from (3.11) and (3.12). This means d is $P-$commuting map on I.

□

4 Conclusion

We have extended Posner-type commutativity results to the setting of symmetric bi-semiderivations on rings containing a semiprime ideal, without imposing global primeness or semiprimeness assumptions. Our analysis demonstrates that centralizing and commuting conditions on such mappings impose stringent structural constraints, ensuring that they are necessarily P-commuting on the ideal. This unifies and generalizes previous results on derivations, semiderivations, and symmetric bi-derivations, highlighting the fundamental influence of semiprime ideals in controlling ring structure. These findings open new avenues for investigating generalized derivations in non-prime rings, with potential applications to Lie ideals, higher-order derivations, and the broader theory of noncommutative algebra.

Acknowledgements The material in this work is a part of first author's Master's Thesis which is supervised by Prof. Dr. Öznur Gölbaşı.

References

1. Alnoghashi, H. M.; Naji, S.; Rehman, N. U. – *On multiplicative (generalized)-derivation involving semiprime ideals*, J. Math., (2023), 1-7.
2. Argaç, N. – *On prime and semiprime rings with derivations*, Algebra Colloq., **13** (2006), no. 3, 371-380.
3. Bedir, Z. – *Some certain differantial identities in semiprime rings with bi-derivation*, Int. J. Open Problems Compt. Math., **17** (2024), no. 4, 66-74.
4. Bergen, J. – *Derivations in prime rings*, Canad. Math. Bull., **26** (1983), no. 3, 267-270.
5. Divinsky, N. J. – *On commuting automorphisms of rings*, Trans. Roy. Soc. Canada Sect. III., **49** (1955), 19-22.
6. Luh, J. – *A note on commuting automorphisms of rings*, Amer. Math. Monthly, **77** (1970), 61-62.
7. Maksa, Gy. – *A remark on symmetric biadditive functions having nonnegative diagonalization*, Glasnik Mat. Ser. III, **15 (35)** (1980), no. 2, 279-282.
8. Mayne, J. H. – *Centralizing automorphisms of prime rings*, Canad. Math. Bull., **19** (1976), no. 1, 113-115.
9. Posner, E. C. – *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.
10. Shujat, F. – *Symmetric generalized biderivations on prime rings*, Bol. Soc. Parana. Mat. (3), **39** (2021), no. 4, 65-72.
11. Söğütçü, E. K.; Gölbaşı, Ö.; Dhara, B. – *On symmetric bi-derivations acting upon prime ideals in any rings*, Ukrainian Math. J., **77** (2025), no. 11, 1509-1523.
12. Vukman, J. – *Symmetric bi-derivations on prime and semi-prime rings*, Aequationes Math., **38** (1989), no. 2-3, 245-254.
13. Yilmaz, D.; Yazarli, H. – *On Lie ideals and symmetric bi-semiderivations in prime rings*, J. Algebra Relat. Topics, **11** (2023), no. 1, 137-146.

Received: 04.III.2026 / Revised: 04.V.2026 / Accepted: 05.V.2026

Authors

Merve Kiliçer,
Department of Mathematics,
Faculty of Science,
Sivas Cumhuriyet University,
Sivas, Türkiye,
E-mail: kilicermerve0@gmail.com

Öznur Gölbaşı (Corresponding author),
Department of Mathematics,
Faculty of Science,
Sivas Cumhuriyet University,
Sivas, Türkiye,
E-mail: ogolbasi@cumhuriyet.edu.tr