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r-Near topologies on nearness approximation spaces

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Abstract. In this manuscript, it is aimed to convert the topology on a set X which is on a nearness approximation space to new set families via indiscernibility relation. Then, if the open sets of the present topology are defined as the set of related elements, the set families, which have weakly related elements, will be obtained. Finally, the topological properties and concepts that these new families hold will be examined.

Keywords: Near set, near topology, topology

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Even though the nearness concept is mentioned in earlier literature, near set concept is first given in the article "Near sets. Special theory about nearness of objects" by J. Peters in 2007. In this study, Peters construct the Nearness Approximation Space by using indiscernibility relation that he constructed via the functions representing object features. Following Peters article, many researchers applied this new set theory to different areas of mathematics ([1–10]) based on Nearness Approximation Space.

On the other hand, the nearness concept in mathematics was first encountered in the concept of neighborhood. Neighborhood is roughly defined as \the set of the elements which are close to point with a certain distance". Outside of the conventional mathematics, it is also possible to refer to nearness by using the distance between two points in any metric space. In topological spaces, the concept of open sets is a natural way of defining the nearness of points. This nearness concept is the state of being the element of the same open set without being bound to a distance independent of the metric spaces. It can briefly be said that the nearness concept is a fundamental approximation of construction of topological spaces.

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In this manuscript, it is aimed to convert the topology on a set X which is on a nearness approximation space to new set families via indiscernibility relation. Then, the set families, which have weakly related elements, will be obtained when the open sets of the present topology are defined as the set of related elements. Finally, the topological properties and concepts, these new families hold, will be examined.

1. Introduction

In this section, we give properties and some definitions of near sets defined by Peters [5, 6].

Object Description

Object Description	
Symbol	Interpretation
$\mathbb{R}$	Set of Real Numbers
O	Set of perceptual objects,
X	set of sample objects,
x	$x \in O$ sample objects,
F	A set of functions representing object features,
B	$B \subseteq F$
Φ	$\Phi : O \to R^L$, object description,
L	is a description length,
i	$i \leq L$
φ_i	$\varphi_i : X \to \mathbb{R}$ probe function, where $\varphi_i \in B$.
$\Phi(x)$	$\Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$

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Objects are known with their definitions. For instance, when we say the definition of $x \in O$ object, we mean $\Phi(x)$ functional value sequence. The important thing here is the selection of necessary probe function $\varphi_i : O \to \mathbb{R}$ to define object. Here, the less the given functions quantity the weaker the definition of the object. We can give the closest definition of the object by multiplying the functions. Let $B \subseteq F$ be the set of functions which presents the description of the object. Then φ_i functions over set B give us a close definition about the object. So, if we define a vector valued function such as $\Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$, this will be a basis for the definition of the object.

Nearness of Object

Relation and Partition Symbols	
Symbol	Interpretation
$\sim_B$	$\sim_B = \{(x, x') : \varphi(x) = \varphi(x'), \forall \varphi \in B\}$, indiscernibility relation,
$[x]_B$	$[x]_B = \{x' \in X : x \sim_B x'\}$ elemantary class,
$O/\sim_B$	$O/\sim_B = \{[x]_B : x \in O\}$ quotient set,
Δ_{φ_i}	$\Delta_{\varphi_i} = \lvert\varphi_i(x) - \varphi_i(x')\rvert$, probe function difference.

Definition 1. Let $x, x' \in O$ and $B \subseteq F$. Then

$$\sim_B = \{(x, x') \in O \times O : \Delta_{\varphi_i} = 0, \forall \varphi_i \in B\}$$

is called the indiscernibility relation on O, where the description length $i \leq |\Phi|$.

In the equivalence relation given in the above, it can be said that all the definitions of the objects x and x' in the same equivalence class are the same, that is, the objects are very near to each other or even the same.

Definition 2. [5] Let $B \subseteq F$ be a set of functions representing the features of objects $x, x' \in O$. Objects $x, x\prime$ are called minimally near each other if there exists $\varphi_i \in B$ such that $x \sim_{\{\varphi_i\}} x'$, $\Delta_{\varphi_i} = 0$. We call it the Nearness Description Principle (NDP).

Fundamental Approximation space

Definition 3. [5] Let O be a set of perceptual objects, F be a set of functions representing object features, $B \subseteq F$ and $\sim_B$be indiscernibility relation. Then the triple $FAS = (O, F, \sim_B)$ is called Fundamental Approximation Space.

Definition 4. [5] Let $(O, F, \sim_B)$ be a Fundamental Approximation Space and $A \subseteq O$.

(1) The set of union of $[x]_B \in O/\sim_B$ which is subset of A, is called B lower approximation of A and denoted as

$$B_*A = \bigcup_{[x]_B \subseteq A} [x]_B.$$

(2) The set of union of $[x]_B \in O/\sim_B$ elements, whose intersection with A is non-empty, is called B upper approximation of A and defined as

$$B^*A = \bigcup_{[x]_B \cap A \neq \varnothing} [x]_B.$$

(3) The boundary of A is denoted as Bnd_BA and defined as

$Bnd_BA = B^*A \backslash B_*A = \{x \in O : x \in B^*A$ and $x \notin B_*A\}$.

Nearness Approximation Space

Nearness Approximation Space Symbols	
Symbol	Interpretation
B	$B \subseteq F$
B_r	$r \leq \lvert B\rvert$ probe functions in B,
$\sim_{B_r}$	B_r Indiscernibility relation defined using B_r,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in X : x \sim_{B_r} x'\}$, equivalence class,
$O/\sim_{B_r}$	$O/\sim_{B_r} = \{[x]_{B_r} : x \in O\}$, quotient set,
$N_r(B)$	$N_r(B) = \{O/\sim_{B_r} : B_r \subseteq B\}$, set of all $O/\sim_{B_r}$partitions,
$N_r(B)_*A$	$N_r(B)_*A = \bigcup_{[x]_{B_r} \subseteq A} [x]_{B_r}$, lower approximation,
$N_r(B)^*A$	$N_r(B)^*A = \bigcup_{[x]_{B_r} \cap A \neq \varnothing} [x]_{B_r}$, upper approximation,

Indiscernibility relation can be defined for each subset B_r, such that $B_r \subseteq B \subseteq F$ and $|B_r| = r$. Let us denote that relation with $\sim_{B_r}$. $\sim_{B_r}$can form different decomposition for each r over O. Here, $\sim_{B_r}$separate O to $[x]_{B_r}$ nearness classes and $O/\sim_{B_r} = \{[x]_{B_r} : x \in O\}$ set is quotient set. Consequently, $N_r(B) = \{O/\sim_{B_r} : B_r \subseteq B\}$ set of all $O/\sim_{B_r}$partitions is obtained.

Definition 5. [5] Let O be a set of perceptual objects, F be a set of functions representing object features and $B \subseteq F$. Then quadruple of $NAS = (O, F, \sim_{B_r}, N_r(B))$ is called Nearness Approximation Space.

Definition 6. [5] Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space and $A \subseteq O$.

(1) The union of $[x]_{B_r} \in O/\sim_{B_r}$ elements, whose are subset of A, is called B_r lower Approximation of A and defined as

$N_r(B)_*A = \bigcup_{[x]_{B_r} \subseteq A} [x]_{B_r}$.

(2) The union of $[x]_{B_r} \in O/\sim_{B_r}$ elements, whose intersection with A is non-empty, is called B_r upper Approximation of A and defined as A

$N_r(B)^*A = \bigcup_{[x]_{B_r} \cap A \neq \varnothing} [x]_{B_r}$.

The Rough Set theory proposed by Pawlak consists of the upper and lower approximation of a set and is obtained with the help of a relation that matches the properties of the object called indiscernibility relation. The near set theory is a more general version of the mentioned set theory. This theory is also based on the indiscernibility relation, but here the indiscernibility relation is considered not for all the functions that give the object properties but separately for each subset. In this way, the partitions of the set of objects are obtained. According to the number of functions (r), $N_r(B)$ sets are formed by the combination of these partitions. Similar to the rough set theory, lower and upper approximations are obtained based on the $N_r(B)$ set. Therefore, unlike the equality in the rough set theory, this system, created in the near sets, is to characterize the number of common properties of objects. The increase in the number of these common properties increases the nearness of objects to each other. Also, if we use all the functions that give the object properties, the approaches in the near set and those in the rough set coincide.

2. r-Near topological spaces

Definition 7. Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$ and (X, τ) be a topological space. The family $\tau_r^* = \{N_r(B)^*(G) : G \in \tau\}$ is called r-near topology which generated by family $(O, F, \sim_{B_r}, N_r(B))$. The elements $N_r(B)^*(G)$ are called r-near open sets. Complement of r-near open sets are called r-near closed sets.

Throughout the paper, $N_r(B)^*(G)$ r-near open sets are denoted as G_r^* for simplicity unless specified otherwise. The family of all r-near closed sets in τ_r^* is denoted as τ_r^{*k}.

Example 1. $X = \{a, b, c, d, e, f\}$, $B = \{\varphi_1, \varphi_2, \varphi_3\}$ a set of functions representing object features and $\tau = \{\varnothing, X, \{c\}, \{a, b, c\}, \{c, d, e, f\}\}$. Then function φ_i defined as

	a	b	c	d	e	f
φ_1	0	1	0	1	0	1
φ_2	0	1	2	0	1	1
φ_3	0	1	2	3	0	1.

Let us construct the equivalence classes for each r-combination. These equivalence classes are formed as

$[a]_{\{\varphi_1\}} = \{a, c, e\}$, $[b]_{\{\varphi_1\}} = \{b, d, f\}$

$[a]_{\{\varphi_2\}} = \{a, d\}$, $[b]_{\{\varphi_2\}} = \{b, e, f\}$, $[c]_{\{\varphi_2\}} = \{c\}$

$[a]_{\{\varphi_3\}} = \{a, e\}$, $[b]_{\{\varphi_3\}} = \{b, f\}$, $[c]_{\{\varphi_3\}} = \{c\}$, $[d]_{\{\varphi_3\}} = \{d\}$,

$[a]_{\{\varphi_1,\varphi_2\}} = \{a\}$, $[b]_{\{\varphi_1,\varphi_2\}} = \{b, f\}$, $[c]_{\{\varphi_1,\varphi_2\}} = \{c\}$, $[d]_{\{\varphi_1,\varphi_2\}} = \{d\}$, $[e]_{\{\varphi_1,\varphi_2\}} = \{e\}$

$[a]_{\{\varphi_1,\varphi_3\}} = \{a, e\}$, $[b]_{\{\varphi_1,\varphi_3\}} = \{b, f\}$, $[c]_{\{\varphi_1,\varphi_3\}} = \{c\}$, $[d]_{\{\varphi_1,\varphi_3\}} = \{d\}$

$[a]_{\{\varphi_2,\varphi_3\}} = \{a\}$, $[b]_{\{\varphi_2,\varphi_3\}} = \{b, f\}$, $[c]_{\{\varphi_2,\varphi_3\}} = \{c\}$, $[d]_{\{\varphi_2,\varphi_3\}} = \{d\}$, $[e]_{\{\varphi_2,\varphi_3\}} = \{e\}$

$[a]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{a\}$, $[b]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{b, f\}$, $[c]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{c\}$, $[d]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{d\}$, $[e]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{e\}$.

Where the set of equivalence classes for each r-combination are formed as

$N_1(B) = \{[a]_{\{\varphi_1\}}, [b]_{\{\varphi_1\}}, [a]_{\{\varphi_2\}}, [b]_{\{\varphi_2\}}, [c]_{\{\varphi_2\}}, [a]_{\{\varphi_3\}}, [b]_{\{\varphi_3\}}, [c]_{\{\varphi_3\}}, [d]_{\{\varphi_3\}}\}$

$N_2(B) = \{[a]_{\{\varphi_1,\varphi_2\}}, [b]_{\{\varphi_1,\varphi_2\}}, [c]_{\{\varphi_1,\varphi_2\}}, [d]_{\{\varphi_1,\varphi_2\}}, [e]_{\{\varphi_1,\varphi_2\}}, [a]_{\{\varphi_1,\varphi_3\}}, [b]_{\{\varphi_1,\varphi_3\}}, [c]_{\{\varphi_1,\varphi_3\}}, [d]_{\{\varphi_1,\varphi_3\}}, [a]_{\{\varphi_2,\varphi_3\}}, [b]_{\{\varphi_2,\varphi_3\}}, [c]_{\{\varphi_2,\varphi_3\}}, [d]_{\{\varphi_2,\varphi_3\}}, [e]_{\{\varphi_2,\varphi_3\}}\}$

$N_3(B) = \{[a]_{\{\varphi_1,\varphi_2,\varphi_3\}}, [b]_{\{\varphi_1,\varphi_2,\varphi_3\}}, [c]_{\{\varphi_1,\varphi_2,\varphi_3\}}, [d]_{\{\varphi_1,\varphi_2,\varphi_3\}}, [e]_{\{\varphi_1,\varphi_2,\varphi_3\}}\}$.

Let us now construct the τ_r^* r-near topology. Where

$N_1(B)^*(\varnothing) = \varnothing$	$N_2(B)^*(\varnothing) = \varnothing$	$N_3(B)^*(\varnothing) = \varnothing$
$N_1(B)^*(X) = X$	$N_2(B)^*(X) = X$	$N_3(B)^*(X) = X$
$N_1(B)^*(\{c\}) = \{a, c, e\}$	$N_2(B)^*(\{c\}) = \{c\}$	$N_3(B)^*(\{c\}) = \{c\}$
$N_1(B)^*(\{a, b, c\}) = X$	$N_2(B)^*(\{a, b, c\}) = \{a, b, c, e, f\}$	$N_3(B)^*(\{a, b, c\}) = \{a, b, c, f\}$
$N_1(B)^*(\{c, d, e, f\}) = X$	$N_2(B)^*(\{c, d, e, f\}) = X$	$N_3(B)^*(\{c, d, e, f\}) = X$

and $\tau_1^* = \{\varnothing, X, \{a, c, e\}\}$. Similiarly $\tau_2^* = \{\varnothing, X, \{c\}, \{a, b, c, e, f\}\}$ and $\tau_3^* = \{\varnothing, X, \{c\}, \{a, b, c, f\}, \{b, c, d, e, f\}\}$.

As seen in the Example 2, even-though τ_1^* and τ_2^* families are topology, τ_3^* family is not a topology. Let us show which of the terms of being topology that τ_r^* families satisfy.

Remark 1. Notice that the $N_1(B)$ and $N_2(B)$ families are not partitions of O. However, the $N_3(B)$ family is a partition. This confusion arises from the selection of all probe functions. Moreover, this is the main relationship between the Rough sets and Near sets that Peters mentioned [5].

Theorem 1. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$ and (X, τ) be a topological space. Then $\varnothing$ and X are r-near open sets.*

Proof. This is obvious from $\varnothing_r^* = \varnothing$ and $X_r^* = X$. ■

Theorem 2. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$ and (X, τ) be a topological space. If $(G_i)_r^* \in \tau_r^*$, for all $i \in I$, then $\cup_{i\in I}(G_i)_r^* \in \tau_r^*$,where I is an arbitrary index set.*

Proof. Let $(G_i)_r^* \in \tau_r^*$, for all $i \in I$. Then $(G_i)_r^* = \underset{[x]_{B_r}\cap G_i \neq \varnothing}{\cup} [x]_{B_r}$.

On the other hand,

$$\underset{i\in I}{\cup}(G_i)_r^* = \underset{i\in I}{\cup}\left(\underset{[x]_{B_r}\cap G_i \neq \varnothing}{\cup}[x]_{B_r}\right) = \underset{[x]_{B_r}\cap\left(\underset{i\in I}{\cup} G_i\right)\neq\varnothing}{\cup} [x]_{B_r}.$$

Since τ is a topolgy, $\underset{i\in I}{\cup} G_i \in \tau$, that is $\cup_{i\in I}(G_i)_r^* \in \tau_r^*$. ■

Definition 8. [11] A subclass $\tau^* \subset P(X)$ is called supra topology on X if $X \in \tau^*$ and τ^* closed under arbitrary union.

Corollary 1. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$ and (X, τ) be a topological space. The families τ_r^* are supra topology on X.*

Theorem 3. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $r \leq s \leq |B|$. Then followings are true;*

(1) If $G \subseteq H$, then $G_r^* \subseteq H_r^*$.

(2) If $B_r \subseteq B_s$, then $[x]_{B_s} \subseteq [x]_{B_r}$.

(3) $G \subseteq G_r^*$, for all $G \in \tau$.

(4) $G_s^* \subseteq G_r^*$, for all $G \in \tau$.

(5) If $G \in \tau$ and $x_0 \in G$, then G_r^* is a neighborhood of x_0.

Proof. (1) Let $G \subseteq H$. Then $G_r^* = \underset{[x]_{B_r}\cap G \neq \varnothing}{\cup}[x]_{B_r} \subseteq \underset{[x]_{B_r}\cap H \neq \varnothing}{\cup}[x]_{B_r} = H_r^*$.

(2)
$$\begin{aligned}[x]_{B_r} &= \{x' \in X : x \backsim_{B_r} x'\}\\ &= \{x' \in X : \{(x, x') : \varphi(x) = \varphi(x'),\ \forall\varphi \in B_r\}\\ &\subseteq \{x' \in X : \{(x, x') : \varphi(x) = \varphi(x'),\ \forall\varphi \in B_s\}\\ &= \{x' \in X : x \backsim_{B_s} x'\}\\ &= [x]_{B_s}\end{aligned}$$

(3) $G = \underset{x\in G}{\cup}\{x\} \subseteq \underset{[x]_{B_r}\cap G \neq \varnothing}{\cup}[x]_{B_r} = G_r^*$

(4) $G_r^* = \underset{[x]_{B_r}\cap G \neq \varnothing}{\cup}[x]_{B_r} \subseteq \underset{[x]_{B_s}\cap G \neq \varnothing}{\cup}[x]_{B_s} = G_s^*$

(5) Obvious from (3). ■

Definition 9. Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$ and (X, τ) be a topological space. If there exists a $G_r^* \in \tau_r^*$ such that $x \in G \subseteq G_r^* \subseteq N$, then N is called r-near neighborhood of x. If N r-near open set, then N is called r-near open neighborhood.

The family of all r-near neighborhood of x is denoted $N_r^*(x)$ and the family of all r-near open neighborhood is denoted as $\tau_r^*(x)$.

Example 2. If we consider Nearness Approximation Space in Example 2 and $\tau_2^* = \{\varnothing, X, \{c\}, \{a, b, c, e, f\}\}$. Then $N_2^*(b) = \{\{a, b, c, e, f\}, X\}$ and $\tau_2^*(c) = \{X, \{c\}, \{a, b, c, e, f\}\}$.

Proposition 1. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$ and (X, τ) be a topological space. Then followings are true;*

(1) If $N \in N_r^*(x)$, then $x \in N$.

(2) If $N_1 \in N_r^*(x)$ and $N_1 \subseteq N_2$, then $N_2 \in N_r^*(x)$.

(3) If $N_1, N_2 \in N_r^*(x)$, then $N_1 \cap N_2 \in N_r^*(x)$.

Proof. (1) Let $N \in N_r^*(x)$. Then there exists a $G_r^* \in \tau_r^*$ such that $x \in G \subseteq G_r^* \subseteq N$. Therefore, $x \in N$.

(2) Let $N_1 \in N_r^*(x)$ and $N_1 \subseteq N_2$. Then there exists a $G_r^* \in \tau_r^*$ such that $x \in G \subseteq G_r^* \subseteq N_1 \subseteq N_2$. Therefore, $N_2 \in N_r^*(x)$.

(3) Let $N_1, N_2 \in N_r^*(x)$. Then there exist $G_r^*, H_r^* \in \tau_r^*$ such that $x \in G \subseteq G_r^* \subseteq N_1$ and $x \in H \subseteq H_r^* \subseteq N_2$. Since τ is a topology, $G \cap H \in \tau$, $x \in G \cap H \subseteq (G \cap H)_r^* \subseteq G_r^* \subseteq N_1$ and $x \in G \cap H \subseteq (G \cap H)_r^* \subseteq H_r^* \subseteq N_2$. Consequently, $x \in G \cap H \subseteq (G \cap H)_r^* \subseteq N_1 \cap N_2$. ■

Proposition 2. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$ and (X, τ) be a topological space. If U neighborhood of x, then $N_r(B)^*(U)$ is also a r-near neighborhood of x.*

Proof. Let U be neighborhood of x. Then there exists a $G \in \tau$ such that $x \in G \subseteq U$. On the other hand,

since $G_r^* = N_r(B)^*(G) = \bigcup_{[x]_{B_r} \cap G \neq \varnothing} [x]_{B_r} \overset{G \subseteq U}{\subseteq} \bigcup_{[x]_{B_r} \cap U \neq \varnothing} [x]_{B_r} = N_r(B)^*(U)$, $x \in G \subseteq G_r^* \subseteq N_r(B)^*(U)$. This completes the proof. ■

Theorem 4. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $r \leq s \leq |B|$. If $N \in N_r^*(x_0)$ for all $x_0 \in X$, then $N \in N_s^*(x_0)$.*

Proof. Let $N \in N_r^*(x_0)$. Then there exists a $G_r^* \in \tau_r^*$ such that $x_0 \in G \subseteq G_r^* \subseteq N$. Since $G_s^* \subseteq G_r^*$ by Theorem 2 (4), $x_0 \in G \subseteq G_s^* \subseteq G_r^* \subseteq N$. This completes the proof. ■

Definition 10. Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $A \subseteq X$.

(1) The set of $cl_r^* A = \cap\{F_r^* : F_r^*$ is a r-near closed set and $A \subseteq F_r^*\}$ is called r-near closure of A.

(2) The set of $int_r^* A = \cup\{G_r^* : G_r^*$ is a r-near open set and $G_r^* \subseteq A\}$ is called r-near interior of A.

Example 3. Let us consider the Nearness Approximation Space and $\tau_3^* = \{\varnothing, X, \{c\}, \{a, b, c, f\}, \{b, c, d, e, f\}\}$ in Example 2. If $A = \{a, b, c, d, f\}$, then the interior of A is $int_r^* A = \{a, b, c, f\}$ and the closure of A is $cl_r^* A = X$.

Theorem 5. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $A, B \subseteq X$. Then the followings are true;*

(1) If $A \subseteq B$, then $cl_r^* A \subseteq cl_r^* B$.

(2) If $A \subseteq B$, then $int_r^* A \subseteq int_r^* B$.

Proof. (1)

$$\begin{aligned} A \subseteq B &\Rightarrow \{F_r^* : F_r^* \in \tau_r^{*k}, B \subseteq F_r^*\} \\ &\subseteq \{F_r^* : F_r^* \in \tau_r^{*k}, A \subseteq F_r^*\} \\ &\Rightarrow \cap\{F_r^* : F_r^* \in \tau_r^{*k}, A \subseteq F_r^*\} \\ &\subseteq \cap\{F_r^* : F_r^* \in \tau_r^{*k}, B \subseteq F_r^*\} \\ &\Rightarrow cl_r^* A \subseteq cl_r^* B \end{aligned}$$

(2)

$$\begin{aligned} A \subseteq B &\Rightarrow \{G_r^* : G_r^* \in \tau_r^*, G_r^* \subseteq A\} \\ &\subseteq \{G_r^* : G_r^* \in \tau_r^*, G_r^* \subseteq B\} \\ &\Rightarrow \cup\{G_r^* : G_r^* \in \tau_r^*, G_r^* \subseteq A\} \\ &\subseteq \cup\{G_r^* : G_r^* \in \tau_r^*, G_r^* \subseteq B\} \\ &\Rightarrow int_r^* A \subseteq int_r^* B \end{aligned}$$

■

Proposition 3. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $A \subseteq X$. Then the followings are true;*

(1) $X \backslash int_r^* A = cl_r^*(X \backslash A)$

(2) $X \backslash cl_r^* A = int_r^*(X \backslash A)$

Proof. (1)

$$\begin{aligned} &X \backslash int_r^* A \\ &= X \backslash \cup \{G_r^* : G_r^* \text{is } r\text{-near open set and } G_r^* \subseteq A\} \\ &= \cap\{X \backslash G_r^* : G_r^* \ r\text{-near open set and } G_r^* \subseteq A\} \\ &= \cap\{F_r^* : F_r^* \ r\text{-near closed set and } X \backslash A \subseteq F_r^*\} \\ &= cl_r^*(X \backslash A) \end{aligned}$$

(2) The proof is similar to proof (1). ■

Theorem 6. *Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $A \subseteq X$. $x \in Cl_r^* A$ if and only if $N \cap A \neq \varnothing$ for all $N \in N_r^*(x)$.*

Proof. Let $x \in Cl_r^* A$. Then $x \in \cap\{F_r^* : F_r^*$ r-near closed set and $A \subseteq F_r^*\}$. Suppose that $N \cap A = \varnothing$ for $N \in N_r^*(x)$. Since $N \in N_r^*(x)$, then there exists a $G_r^* \in \tau_r^*$ such that $x \in G \subseteq G_r^* \subseteq N$. Hence $G_r^* \cap A = \varnothing$. If we choose $F_r^* = X \backslash G_r^*$, then $A \subseteq F_r^*$ and $x \notin F_r^*$. Consequently, $x \notin \cap\{F_r^* : F_r^*$ r-near closed set and $A \subseteq F_r^*\}$. This is contradiction.

Conversely, let $N \cap A \neq \varnothing$ for all $N \in N_r^*(x)$. Suppose that $x \notin Cl_r^* A$. Then $x \notin \cap\{F_r^* : F_r^*$ r-near closed set and $A \subseteq F_r^*\}$. Therefore, there exists a F_r^* r-near closed set include A such that $x \notin F_r^*$. If we choose $G_r^* = X \backslash F_r^*$, $G_r^* \in \tau_r^*$ and $G_r^* \in N_r^*(x)$. Moreover, since $A \subseteq F_r^*$, $A \cap G_r^* = \varnothing$. This is contradiction. ■

Definition 11. Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $A \subseteq X$. The set of $CoreA = \{x \in X : x \in G \subseteq G_r^* \subseteq A$ ve $G \in \tau\}$ is called core of A.

Proposition 4. *$(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and $A \subseteq X$. Then the followings are true;*

(1) $CoreA \subseteq int_r^* A$

(2) $int_r^* A \subseteq intA$

(3) $clA \subseteq cl_r^* A$

Proof. (1)

$x \in CoreA$

$\Rightarrow x \in \{x \in X : x \in G \subseteq G_r^* \subseteq A$ ve $G \in \tau\}$

$\Rightarrow$ Then there exist $G \in \tau$ such that $x \in G \subseteq G_r^* \subseteq A$.

$\Rightarrow$ Then there exist $G_r^* \in \tau_r^*$ such that $x \in G_r^* \subseteq A$.

$\Rightarrow x \in int_r^* A$

(2) Let $x \in int_r^* A$. Suppose that $x \notin intA$. Then $x \notin \cup\{G \in \tau : G \subseteq A\}$. Therefore $G \nsubseteq A$ for all $x \in G \in \tau$. Thus $G_r^* \nsubseteq A$ for all $x \in G_r^* \in \tau_r^*$. Consequently, $x \notin int_r^* A$. This is contradiction.

(3) Let $x \in clA$. Suppose that $x \notin cl_r^* A$. Then there exist $N \in N_r^*(x)$ such that $N \cap A = \varnothing$. Since $N \in N_r^*(x)$, then there exist $G \in \tau$ such that $x \in G \subseteq G_r^* \subseteq N$. Therefore, $N \in N(x)$ and $N \cap A = \varnothing$. This is contradiction. ■

3. Sequences

Definition 12. Let $(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and (x_n) be a sequence in X. The sequence (x_n) convergence to x_0 and, denoted $x_n \xrightarrow{r} x_0$, if and only if each $N \in N_r^*(x_0)$**,** there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $x_n \in N$**.**

Example 4. Let $X = \{a, b, c, d, e, f\}$, $B = \{\varphi_1, \varphi_2, \varphi_3\}$ be a set of functions representing object features and $\tau = \{\varnothing, X, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. Then the function φ_i defined as

	a	b	c	d	e
φ_1	0	1	0	2	2
φ_2	0	1	0	1	1
φ_3	0	2	0	1	2

Let us construct the equivalence classes for each r-combination. These equivalence classes are formed as

$[a]_{\{\varphi_1\}} = \{a, c\}$, $[b]_{\{\varphi_1\}} = \{b\}$, $[d]_{\{\varphi_1\}} = \{d, e\}$

$[a]_{\{\varphi_2\}} = \{a, c\}$, $[b]_{\{\varphi_2\}} = \{b, d, e\}$

$[a]_{\{\varphi_3\}} = \{a, c\}$, $[b]_{\{\varphi_3\}} = \{b, e\}$, $[d]_{\{\varphi_3\}} = \{d\}$,

$[a]_{\{\varphi_1,\varphi_2\}} = \{a, c\}$, $[b]_{\{\varphi_1,\varphi_2\}} = \{b\}$,$[d]_{\{\varphi_1,\varphi_2\}} = \{d, e\}$

$[a]_{\{\varphi_1,\varphi_3\}} = \{a, c\}$, $[b]_{\{\varphi_1,\varphi_3\}} = \{b\}$, $[d]_{\{\varphi_1,\varphi_3\}} = \{d\}$, $[e]_{\{\varphi_1,\varphi_3\}} = \{e\}$

$[a]_{\{\varphi_2,\varphi_3\}} = \{a, c\}$, $[b]_{\{\varphi_2,\varphi_3\}} = \{b, e\}$, $[d]_{\{\varphi_2,\varphi_3\}} = \{d\}$

$[a]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{a, c\}$, $[b]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{b\}$, $[d]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{d\}$, $[e]_{\{\varphi_1,\varphi_2,\varphi_3\}} = \{e\}$.

Then $\tau_1^* = \{\varnothing, X, \{b, d, e\}\}$. Similiarly $\tau_2^* = \{\varnothing, X, \{b, e\}, \{a, b, c, e\}, \{a, c, d, e\}\}$ and $\tau_3^* = \{\varnothing, X, \{b\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$ and on the other hand

$N_1^*(a) = \{X\}$, $N_2^*(b) = \{X, \{b, d, e\}\}$, $N_1^*(c) = \{X\}$, $N_1^*(d) = \{X\}$, $N_1^*(e) = \{X\}$

$N_2^*(a) = \{\{a, b, c, e\}, X\}$, $N_2^*(b) = \{U \subseteq X : \{b, e\} \subseteq U\}$, $N_2^*(c) = \{X\}$, $N_2^*(d) = \{\{a, c, d, e\}, X\}$, $N_2^*(e) = \{X\}$

$N_3^*(a) = \{U \subseteq X : \{a, b, c\} \subseteq U\}$, $N_3^*(b) = \{U \subseteq X : \{b\} \subseteq U\}$, $N_3^*(c) = \{U \subseteq X : \{a, c, d\} \subseteq U\}$, $N_4^*(d) = \{U \subseteq X : \{a, c, d\} \subseteq U\}$, $N_2^*(e) = \{X\}$.

The convergences of $(x_n) = (a, b, c, c, b, a, b, a, b, a, ...)$ for the topologies above are

$x_n \to a$	$x_n \overset{1}{\to} a$	$x_n \overset{2}{\to} a$	$x_n \overset{3}{\to} a$
$x_n \nrightarrow b$	$x_n \overset{1}{\nrightarrow} b$	$x_n \overset{2}{\nrightarrow} b$	$x_n \overset{3}{\nrightarrow} b$
$x_n \nrightarrow c$	$x_n \overset{1}{\to} c$	$x_n \overset{2}{\to} c$	$x_n \overset{3}{\nrightarrow} c$
$x_n \nrightarrow d$	$x_n \overset{1}{\to} d$	$x_n \overset{2}{\nrightarrow} d$	$x_n \overset{3}{\nrightarrow} d$
$x_n \to e$	$x_n \overset{1}{\to} e$	$x_n \overset{2}{\to} e$	$x_n \overset{3}{\to} e$.

If we analyze the example above, it can be seen that there is a relationship between the convergence of (x_n) on τ and on r-near topologies. Let us give this relationship with the following theorem.

Theorem 7. *$(O, F, \sim_{B_r}, N_r(B))$ be a Nearness Approximation Space, $X \subseteq O$, (X, τ) be a topological space and (x_n) be a sequence in X. Then followings are true;*

(1) If $x_n \xrightarrow{s} x_0$, then $x_n \xrightarrow{r} x_0$, for all $r \leq s$.

(2) If $x_n \to x_0$, then $x_n \xrightarrow{r} x_0$, for all $r \leq |B|$.

Proof. (1) Let $r \leq s$ and $x_n \xrightarrow{s} x_0$. Then there exist $n_0 \in \mathbb{N}$ for all $N \in N_s^*(x_0)$ such that $x_n \in N$ for all $n \geq n_0$. By theorem 2, for all $N \in N_s^*(x_0)$,there exist $n_1 \in \mathbb{N}$ such that $(x_n) \in N$ for all $n \geq n_1$. Therefore $x_n \xrightarrow{r} x_0$.

(2) Let $x_n \to x_0$ and $r \leq |B|$. Then there exist a $n_0 \in \mathbb{N}$ for all $N \in N(x_0)$ neighborhood such that $x_n \in N$ for all $n \geq n_0$. Therefore, since all G open set, which contains x_0, is a neighborhood, there exist an $n_1 \in \mathbb{N}$ integer such that $x_n \in G$ for all $n \geq n_1$. Consequently, there exist an $n_1 \in \mathbb{N}$ integer for all $N \in N_s^*(x_0)$ r-near neighborhood such that $x_n \in G \subseteq G_r^* \subseteq N$ for all $n \geq n_1$. ■

Conclusion 1. This article is the first study to investigate topological structures on near sets. The obtained results will be a reliable reference to the future researchs.

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Compliance with Ethical Standards:

Conflict of Interest: I (Serkan ATMACA) declare that I have no conflict of interest.

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